Article

The interpretant signs produced in mathematical modelling activities[1]

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Abstract
In this paper we investigate the interpretant signs production during mathematical modelling activities development. The theoretical framework is based on the relationship between mathematical modelling and some elements of the semiotics structured by Charles Sanders Peirce, more specifically in his interpretant theory. This theoretical framework is associated with an empirical research in which modelling activities are developed by students of a Degree in Mathematics in a differential and integral calculus subject. The analysis of the activities follows qualitative research directions and leads us to infer that the students produced different interpretant signs when they developed mathematical modelling activities. In the warm up activity development it was possible to identify immediate interpretant signs, while during the follow up activities the students produced dynamic interpretants and some final interpretants for the extreme function values. In general, the interpretant signs give evidence of how the derivative was being used by the students to decide on the existence of maximum or minimum values of a function obtained when they developed mathematical modeling activities.

Keywords: Mathematics education, Mathematical modeling, Peircean semiotics.

Introduction

The construction of knowledge has been discussed in research of different areas at the national and international levels. Regarding Mathematics Education, there is an understanding that this construction is related to the activities developed by students and the possibilities they provide for addressing what is being taught (ALMEIDA; SILVA, 2017; ÄRLEBÄCK; DOERR, 2015; among others).

[1] This article includes part of the research of the first author's master thesis, under the guidance of the second author. Part of these results has already been presented in an event in the research area, and this text is an expanded version in which new discussions and deepening have been carried out.

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In this context, we understand mathematical modeling as an approach that makes it possible to investigate non-mathematical situations through mathematics (ALMEIDA; BRITO, 2005).

As discussed in Almeida e Silva (2017), the approach to these situations refers to the use of different representations and this use can be associated with knowledge construction, especially in the school context. The use of representations may indicate how knowledge about mathematical objects is constructed by the subjects. The approach to these representations refers to the theoretical contribution of semiotics, specifically with regard to the study of signs.

In this text, we articulate the mathematical modeling in Mathematical Education and elements of semiotics structured by Charles Sanders Peirce and recognized in the literature as Peirce's semiotics, focusing on what Peirce (2015) calls interpreters' theory with the aim, from this articulation, to investigate the use and production of interpreting signs. These signs are classified in Peirce (1972) as immediate interpreters, dynamic interpreters and final interpreters. Our look is directed at interpreters produced by the students of a Mathematics Degree course in the development of a sequence of mathematical modeling activities that require the use of the concept of maximum and minimum functions.

Mathematical Modeling in Mathematical Education

In general, mathematical modeling activities have as a starting point an initial situation (problematic) and as a final situation (solution to the initial situation), as discussed by Almeida (2010) and make it possible to explore situations originating outside the context of mathematics.

The introduction of mathematical modeling activities in math classes therefore has the challenge of "recreating culturally distinct environments in the classroom from the school context" (CARREIRA; BAIOA, 2018, p. 201). However, these activities also have the purpose of providing the introduction, discussion and systematization of mathematical contents. In this sense, mathematical modeling can be understood as a pedagogical alternative for the teaching of mathematics (ALMEIDA; BRITO, 2005).

In the field of Mathematics Education, Lesh et al. (2003) ponder that when there is the intention of teaching mathematics through mathematical modeling activities, isolated activities are rarely sufficient to produce the expected results (related to the learning of mathematical concepts), which justifies the importance of developing sequences of structurally related activities.

In order to indicate how activities can be organized to characterize the structure of a sequence of mathematical modeling activities, Lesh et al. (2003) present an organizational scheme as shown in Figure 1.

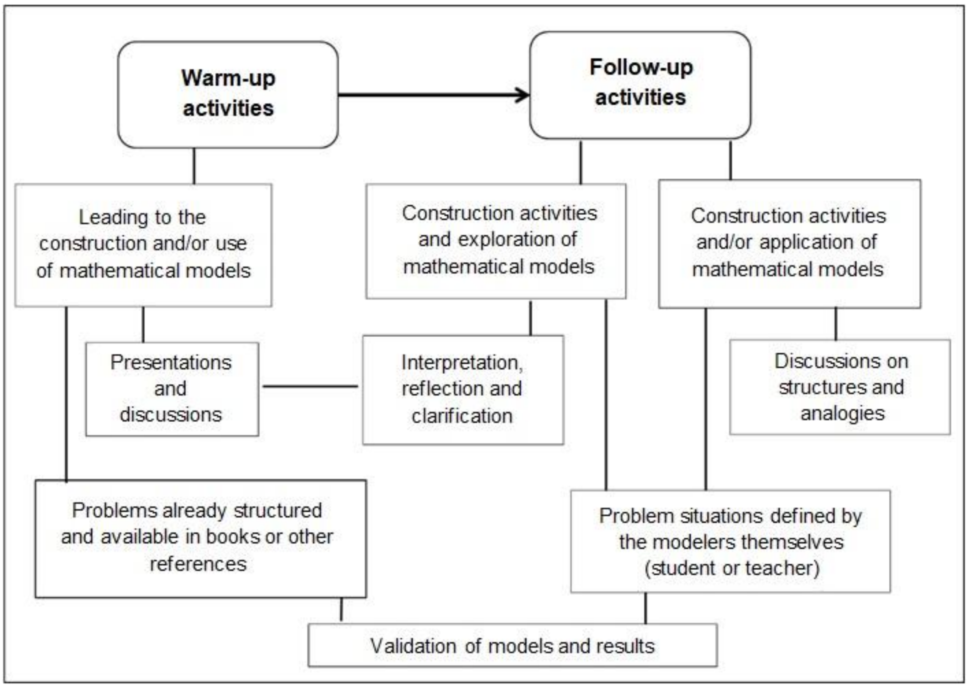


Figure 1 - Organizational scheme for modeling activity sequences
Source: Adapted from Lesh et al. (2003, p. 45).

The first activities developed with the students, the so-called warm-up activities, aim at proposing to the student's problem situations that lead to the construction or use of a mathematical model. These are situations that have already become sturdy in which a problem is solved through the construction, use and analysis of a mathematical model. These activities enable the teacher to answer some questions about minimum prerequisites for students to begin developing mathematical modeling activities (LESH et al., 2003).

In these activities, students are encouraged to work in teams and are often used by teachers at the beginning of a course unit. One of their objectives is that the student reveals to the teacher possible conceptual weaknesses so that they can be explored during the classes (LESH et al., 2003).

After the development of warm-up activities, according to Lesh et al. (2003), it is important to conduct dynamic presentations and discussions with the whole class in order to enable students to have contact with other ways of thinking, discuss the strengths and weaknesses of alternative approaches and identify directions for improvement in their own work or in the work of others.

Follow-up activities, in turn, consist of activities that should help students recognize connections between their theoretical knowledge and everyday situations. Thus, in addition to building models, students need to explore and apply models as defined by Ärlebäck and Doerr (2015).

For the authors, a model exploration activity lends itself to getting the student to explore the mathematical structure of a model derived from a problem situation. In view of this, the focus of this type of activity is on the mathematical structure underlying the mathematical model.

Exploration activities of mathematical models should lead students to contrast strengths and weaknesses of different models and use mathematical language accurately, carefully and according to the problem situation under study (ÄRLEBÄCK; DOERR, 2018).

By developing model application and exploration activities, discussions about structural similarities between the activities will foster students' experiences and, in addition to thinking about the models constructed, they will also become able to discuss the use of mathematical models.

Underlying familiarity with mathematical language is familiarity with mathematical modeling, mathematical concepts and procedures. In this context, it also becomes relevant for students to appropriate representations that allow them to deal with the mathematical objects that emerge from mathematical modeling activities. In order to direct our attention to this aspect, we look at the interpretative signs produced by the students during the development of a sequence of mathematical modeling activities.

Peircean Semiotics

Charles Sanders Peirce (1839-1914) mathematician, physicist, astronomer, founded Peircean semiotics and used the term Semeiotic from the logic conceived as a philosophy of language.

For Peirce (1972, p. 94),

> A sign, or representamen, is that which, in a certain aspect or way, represents something for someone. It is addressed to someone, that is, it creates, in that person's mind, an equivalent sign, or perhaps a more developed sign. To the sign, thus created, I call the interpreter of the first sign. The sign represents something, its object. It is placed in the place of this object, not in all aspects, but with reference to a type of idea that I sometimes call the foundation of the represent.

Peirce's definition of a sign brings to light a fundamental characteristic of Peircean semiotics: the triadic relationship that associates a representamen, the "material" part of the sign; an object, that to which it is referred; and an interpreter, that which derives or is generated by an interpreter through the relationship between the representamen and the object (SANTAELLA, 2012).

The generation of new signs - interpreter signs - is associated with the concept of semiosis that is characterized as an evolutionary activity and corresponds to the action proper to the sign of being interpreted by another sign (PEIRCE, 2015). The interpretive sign has the nature of a sign created in an interpreting mind. For Nöth (2008), the cognitive effect of the sign on the interpreter is what enables it to generate new signs and is what characterizes semiosis.

In this context, Drigo (2007) states that semiosis is triggered by the updating of the mind, that is, a new sign is generated with the identification of discomfort or instability, whose overcoming is mediated by semiosis.

This updating of the mind is related to a characteristic of semiosis, which according to Almeida (2010, p. 390), corresponds to "a characteristic process of the human capacity to produce and understand signs of the most diverse natures.

As Almeida e Silva (2018) points out, Peirce dedicated part of his investments in the structuring and classification of interpretative signs. In Peirce (1972), the classification of interpreters is given as: immediate interpreter, dynamic interpreter and final interpreter.

The immediate interpreter refers to the aspect that each sign has its peculiar interpretability, before it reaches any interpreter. It is an abstraction consisting of a possibility of representing something for someone.

The dynamic interpreter is the effect produced in the mind of the interpreter by the sign. It is the effect that the sign determines in that mind.

The final interpreter, in turn, according to Peirce (2015, p 164), "is what would finally be the true interpretation if one considered the matter so deeply that a definitive opinion could be reached".

The interpretative signs in Peirce's theory are means used to represent something for someone, they are means of thinking, understanding, reasoning, learning (SANTAELLA, 2012).

To discuss the relationship between action and the production of signs in mathematical modeling activities and the knowledge of students, Almeida e Silva (2017) rely on aspects of Peircean semiotics, more specifically regarding semiosis, for the analysis of a mathematical modeling activity. From this analysis, the authors infer that semiosis generates new signs that trigger the construction of new knowledge by the interpreters and "in this sense, semiosis represents the characteristic process of the human capacity to produce and understand signs of various kinds" (ALMEIDA; SILVA, 2017, p. 216).

In this text we look at a sequence of mathematical modeling activities respecting the scheme proposed by Lesh et al. (2003), with the purpose of investigating the use and production of interpretative signs in the development of mathematical modeling activities, in which the study of extremes of a function is fundamental, so that the solution of the problem under study can be analyzed.

Methodological aspects

In order to investigate the use and production of interpretative signs in the development of a sequence of mathematical modeling activities, we analyzed three modeling activities developed by a group of three students from the 2nd year of the Mathematics Degree in the subject of Differential and Integral Calculus I, in the first semester of the 2017 school year.

The development of the activities was conducted by the teacher/researcher and one of the authors of this article.

During the development of the activities, data were collected through audio and video recordings. In addition, field journal entries were made by the teacher/researcher, indicating important elements regarding the production and use of signs by students during the development of the activities.

The analysis of the information collected follows the principles of qualitative research following the directions suggested in Bogdan and Biklen (2003).

The interpretative signs in a sequence of mathematical modeling activities

The sequence of activities developed is composed of three mathematical modeling activities, whose themes concern Internet Users in Brazil (A1), Optimization of the size of electrical cables (A2) and Water loss in the water distribution process (A3).

Activity 1: Internet Users in Brazil

The information on the subject was given to the students in two texts. One text referring to the number of Internet users in Brazil and the other one referring to the possible new bug of the millennium foreseen to occur in 2038 due to the configurations of the current operating systems. These texts are reports published in the Federal Government Portal and the Online Magazine Exame.com, respectively.

When contacting the texts, the students had a first impression regarding what they could study. This first contact refers to what is related to chance, to what is not yet a concrete fact, that is, such an impression is, for the students, at this moment, an immediate interpreter.

Regarding the scheme of a sequence of activities, as presented by Lesh et al. (2003), this first activity was developed precisely with the intention of helping students to look mathematically at everyday situations while they become familiar with mathematical modeling activities.

In addition to the texts, students also received a table containing the number of households with Internet access in the country over the past 11 years, according to information made available by PNAD and the World Bank.

From the discussion of the texts and the contact with data on the number of users from 2006 on, two problems of interest were defined, as shown in Figure 2.

Considering the data presented, it was perceived by the students the need to define a hypothesis to deliberate on the number of Internet users in each home. Thus, based on information from the texts, it was established: H1: Each household is composed of a couple and two more children. With this hypothesis, the students converted the data of PNAD, from quantity of domiciles to quantity of users.

DATA:

Year	Users (in millions)
2006	58,68
2007	64,8
2008	75,84
2009	79,24
2010	89,96
2011	96,56
2012	105,44
2013	113,36
2014	121,28
2015	123,88
2016	129,16

PROBLEMS:

1) *How many people in Brazil will have access to the Internet in the year 2038, the year of the possible "millennium bug"?*

2) *What is the maximum number of people in Brazil who will have access to the internet?*

VARIABLES:

n: time (years)

$U(n)$: number of internet users in Brazil in time n.

Figure 2 - Information on activity Internet users in Brazil
Source: students' report.

It is worth mentioning that before the hypothesis elaboration and the variables definition, the students were in doubt regarding several terms presented in the texts such as bug, bit and byte. Therefore, in order to solve such questions, students should carry out complementary research in relation to the subject to be studied.

In this case, the data in Figure 2 also constitute a sign that produces in the students a first impression regarding the phenomenon of number of Internet users in Brazil. As Santaella (2012, p. 46) quotes, "the first apprehension of things, which for us appear, is already a translation, a very thin film between us and the phenomena".

In this first contact the students still do not have in mind to which mathematical object the data refer to, making no relation between the sign and properties or representations of the object.

The discussions in the group then become an immediate interpreter that reveals such impression. The questions, in turn, indicate the interpretability of the sign towards what they could know in relation to the amount of internet users in the country through this data, thus having characteristics of a dynamic interpreter.

Furthermore, we can consider that, since the elaboration of the hypothesis was a reaction to the impressions that the interpreters had when contacting the information, such elaboration is characterized as an action subsequent to the impression of the sign, that is, a dynamic interpreter.

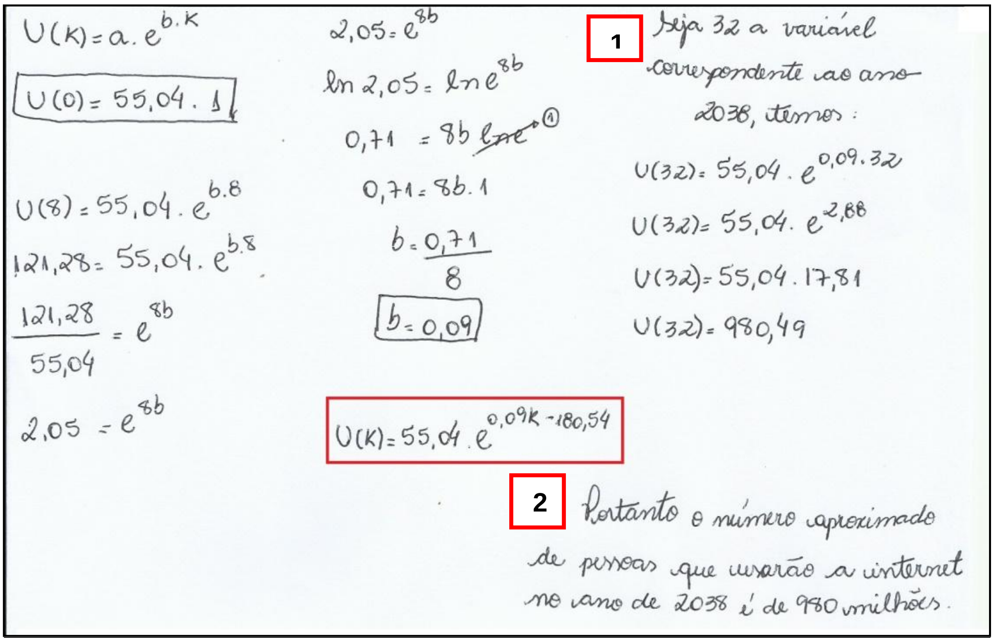
To solve the problems, the students decided to analyze the variation in the number of users over the years (Table 1), thus realizing that such variation followed a certain pattern. Analyzing this pattern, the group inferred that the data grew exponentially, since the ratio between the data were close to constant 1, and so they decided to adjust an exponential function of type U(k)=a.e^(b . k), being U the number of Internet users in Brazil, k the auxiliary variable that represents the time in years and the coefficients a and b the parameters of the function to be adjusted.

Table 1 - Analysis of data behavior

Year	Auxiliary Variable (k)	Number of users	Arithmetic behavior of the data ($k_1 - k$)	Geometric behavior of the data ($\frac{k_1}{k}$)
2006	0	58,68	6,12	1,104294
2007	1	64,8	11,04	1,17037
2008	2	75,84	3,4	1,044831
2009	3	79,24	10,72	1,135285
2010	4	89,96	6,6	1,073366
2011	5	96,56	8,88	1,091964
2012	6	105,44	7,92	1,075114
2013	7	113,36	7,92	1,069866
2014	8	121,28	2,6	1,021438
2015	9	123,88	5,28	1,042622
2016	10	129,16	-	-

Source: students' report.

Once it was decided that an exponential function would be used to describe the number of Internet users over time, the mathematical procedures adopted by the students to adjust this function were referred to in Figure 3:



$U(k) = a.e^{b.k}$

$U(0) = 55,04.1$

$U(8) = 55,04.e^{b.8}$

$121,28 = 55,04.e^{b.8}$

$\frac{121,28}{55,04} = e^{8b}$

$2,05 = e^{8b}$

$2,05 = e^{8b}$

$\ln 2,05 = \ln e^{8b}$

$0,71 = 8b \ln e$ → 1

$0,71 = 8b.1$

$b = \frac{0,71}{8}$

$b = 0,09$

$U(k) = 55,04.e^{0,09k-180,54}$

1 Seja 32 a variável correspondente ao ano 2038, temos:

$U(32) = 55,04.e^{0,09.32}$

$U(32) = 55,04.e^{2,88}$

$U(32) = 55,04.17,81$

$U(32) = 980,49$

2 Portanto o número aproximado de pessoas que usarão a internet no ano de 2038 é de 980 milhões.

Translation:

1. *Being 32 the variable for the year 2038, we have:*
2. *Then, the approximate number of people who will use the Internet in the year 2038 is 980 million.*

Figure 3 - Mathematical procedures adopted by the group
Fonte: students' report.

The first model determined was U(k)=55.04 .e^(0.09.k-180.54), U being the number of households with Internet access in Brazil (in millions) and k the time in years.

Table 2 - Validation of the model $\boldsymbol{U(k) = 55,04.e^{0,09k-180,54}}$

Year	Actual Value	Modelled Value	Difference (Real - Modelled)
2006	58,68	55,04	3,64
2007	64,8	60,22	4,58
2008	75,84	65,89	9,95
2009	79,24	72,10	7,14
2010	89,96	78,89	11,07
2011	96,56	86,32	10,24
2012	105,44	94,45	10,99
2013	113,36	103,34	10,02
2014	121,28	113,08	8,20
2015	123,88	123,72	0,16
2016	129,16	135,38	-6,22

Source: students' report.

Before presenting the answer to the problem of determining the number of Internet users, the groups made the validation of the model (Table 2) and, as the data of the model approached the data presented in the activity, they considered the model appropriate for the situation. Thus, with the model defined, the group concluded that in 2038 there will be more than 980 million Internet users in Brazil.

It is possible to affirm that this is the final interpreter for this situation since it is, at least for this moment, a conclusion that the students attributed to the phenomenon (quantity of Internet users in Brazil in 2038). Moreover, this was obtained from properties and mathematical laws activated by the mind of the interpreters, which is characteristic of the final interpreter.

However, although mathematically the model was adequate, interpreting the model obtained from the phenomenon, the students had doubts regarding the veracity of the answer obtained.

Thus, a growth in the semiosis of the students seems to have been configured influenced by their knowledge, both mathematical and of the phenomenon, which can be inferred considering that the students sought a "level of fidelity", as they call Almeida, Silva and Veronez (2015, p. 10), between the model and the phenomenon in question.

Then, from searches on the Internet, the group had contact with a report published on the G1 Portal which, citing a survey by the Brazilian Institute of Geography and Statistics (IBGE), states that the Brazilian population will never exceed the total of 228.4 million people. Furthermore, the same report mentions that Brazil will have this population in 2042.

Thus, the students concluded that it is not possible to have such a large number of Internet users in Brazil in 2038, since, for them, it is not possible for the Brazilian population to grow so much in such a short time. This analysis is evidenced in the dialogues between the students of the group.

E1: 980 million Internet users in Brazil? Impossible, even more so in 2038.
E3: How many people do they have in Brazil today? Research on the internet.
E1: 207 million.
E3: Gee, you're wrong. Brazil will never have 980 million inhabitants.

From that, the group defined another hypothesis H2: the Brazilian population will stabilize with 230 million inhabitants. After defining this hypothesis, the group sought to adjust an exponential function with asymptotic behavior. At this moment, the students do what Ärlebäck and Doerr (2015) call the expansion of the model. The group uses a model structure already known to them (exponential) and expands it, in this case, inhibiting the growth of the number of users (asyntactic exponential).

This mathematical approach to the situation also allows us to show a growth in the semiosis of students, since their previous experiences with this type of function allowed the generation of new interpretative signs.

It is worth mentioning that all the students in the class had already studied the subject of Functions in the course and, for this reason, already knew the function of the exponential type of asymptotic growth.

The mathematical procedures adopted by the groups for the deduction of a second model are shown in Figure 4.

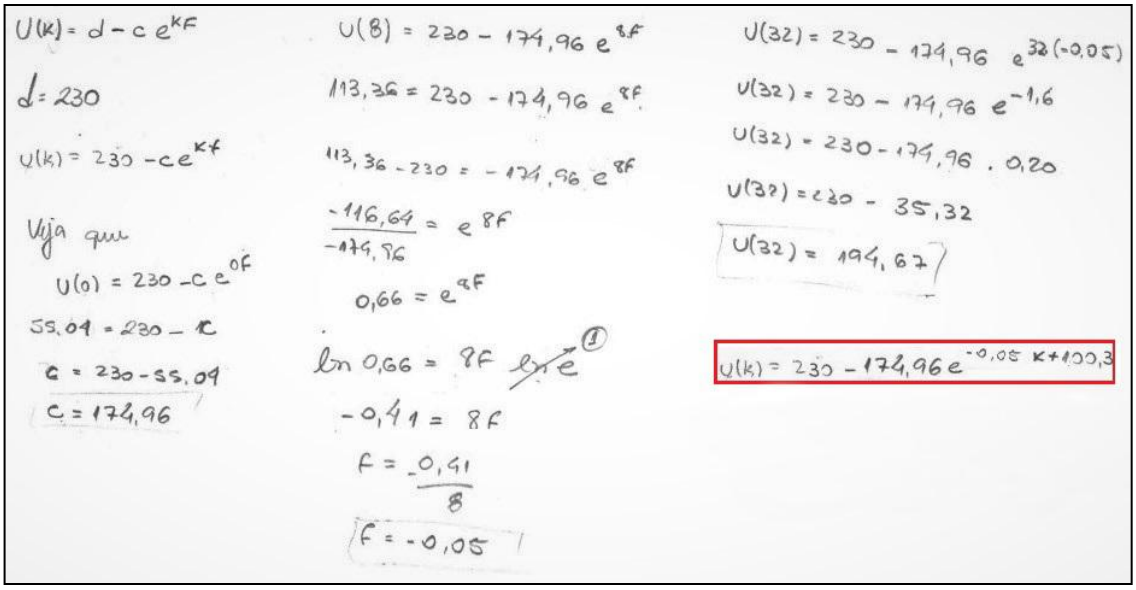


Figure 4 - Mathematical procedures adopted by the groups
Source: students' report.

The second model determined was U(k)=230-174.96 . e^(-0.05.k+100.3), U being the number of Internet users in Brazil (in millions) and k the time (in years).

From the model built, the group concluded that in 2038, 194.67 million Brazilians will have access to the Internet, since U(238)=194.67. Here, the model built and the group's resolution to the situation can be taken as final interpretative signs for the group regarding the investigated phenomenon (number of internet users), since it was what the students considered as definitive regarding the problem.

Also, in order to respond to the second problem situation (what is the maximum number of people who will have access to the Internet in Brazil), the students initially sought to analyze the graph of the defined model, starting in 2006, as shown in Figure 5.

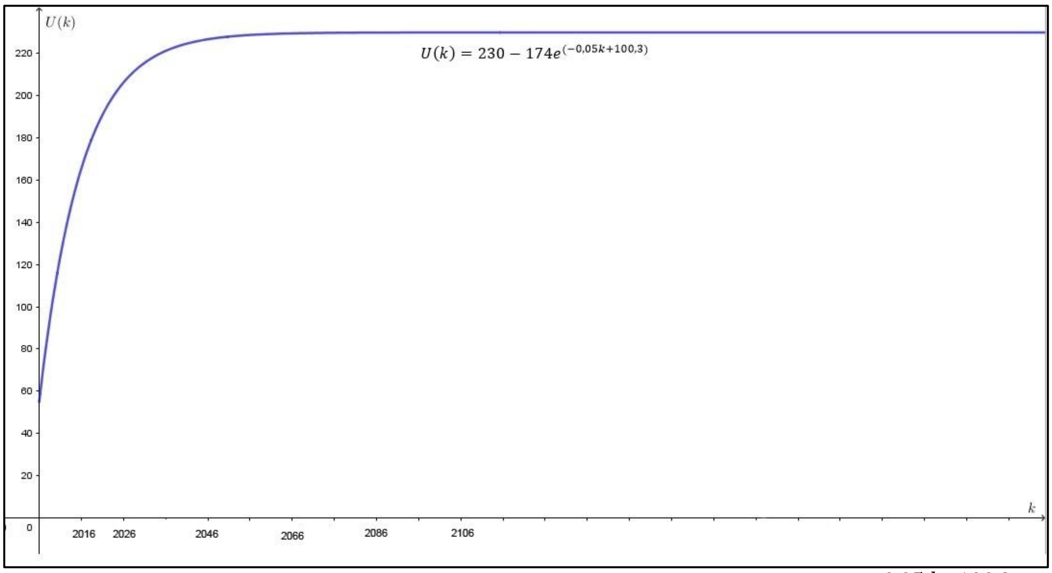


Figure 5 - Graphical representation of model $U(k) = 230 - 174{,}96e^{-0{,}05.k+100{,}3}$
Source: Students' report.

The production of this interpretive sign (graph of the model obtained) reveals the personal experiences of the students in order to understand the situation. However, the visual analysis of the graph was insufficient to determine the maximum number of Internet users, precisely because of the type of function adjusted by the group (asymptotic exponential).

At this moment, discussions have developed in order to investigate mathematical theories that, in some way, would allow the analysis of a function in order to determine its maximum value and also what are the conditions for a function to have maximum value. Thus, discussions began in class regarding the concept of extreme values of a function and how such knowledge could be used to solve the problem under study.

Such discussions, reflections and clarifications are in conformity with what was proposed by Lesh et al. (2003) regarding a warm-up activity in a sequence of mathematical modeling activities. In the case of this sequence, this first activity served as a warm-up activity for the study of the concept of extreme values of a function, since it allowed the students to initiate some reflections regarding the existence of maximum and minimum values of a function. The mathematical model built for the phenomenon of the number of Internet users in Brazil - a function of the exponential type - although asymptotic and indicating that there would be a limit to the number of users, has not yet made it possible to determine the maximum value of a function.

Activity 2: Optimization of the electric cables dimension

When this activity was developed, the students had already studied with the professor the formal definition of the concept of extreme values of a function. Thus, this activity enabled the students to investigate a situation in which the construction and exploration of a mathematical model would be supported precisely by the use of this concept. In this sense, the activity can be characterized as one of the follow-up activities, in reference to the scheme of a sequence of mathematical modeling activities proposed by Lesh et al. (2003).

The theme of interest to students refers to the choice of the energy cable according to the gauge. For the students' integration with the theme, the text "Safe Electrical Installation with Technical Standard ABNT 5410" was discussed, as shown in Figure 6.

In addition to the text, the groups also had access to the data collected by the research professor in a Control and Automation laboratory that simulates the voltage drop of an electrical conductor according to its nominal section, as shown in Figure 6.

In this case, all these materials previously delivered to the students (instructional text and table with the collected data) constitute signs that produce in the students a first impression regarding the phenomenon under study.

From the group's discussion regarding the data, the problem that should be investigated in this activity was defined: For the assembly of an electrical extension (for residential use) what would be the electrical cable gauges that would present the smallest and largest voltage drop?

In this way, the group discussions also become immediate interpretative signs that reveal their impression regarding the situation. The question defined for investigation is what indicates the interpretability of the sign, towards what the students could know regarding the voltage drop by means of the data presented, thus being characterized as a dynamic interpreter.

Introductory text of Activity 2

Safe Electrical Installation with ABNT NBR 5410 Technical Standard

When it comes to electricity, the essential is safety. A safer and higher quality electrical installation is what guarantees the Brazilian Technical Regulatory Standard (NBR) 5410:2004. This Standard establishes the conditions that low voltage electrical installations must meet in order to ensure the safety of people and animals, the proper functioning of the installation and the conservation of property. This NBR 5410 determines, among other things, the nominal section (gauge) of cables, or electrical conductors, to be used in electrical installations. Such section is defined according to the electrical load that will pass through this conductor.
In addition to protection, the choice of the ideal cable can generate savings, since, among other things, it can minimize the waste of energy. In any electrical circuit, the drop of electrical voltage is a factor present. This drop is an anomaly caused by the distances covered by the electric current in a given circuit: the longer the conductor is, the higher the voltage drop, since there is an increase in the electric resistance due to the larger amount of material used to make larger conductors.

Source: Portal of the Brazilian Association of Technical Standards (ABNT).

Voltage drop according to wire gauge

Cable gauge (mm^2)	Voltage drop per meter of cable (Volt - V)
1,5	0,023
2,5	0,017
4,0	0,012
6,0	0,02
10,0	0,095
16,0	0,35
25,0	0,98
35,0	2
50,0	4,5
70,0	9,5

Source: Data collected by the researcher.

Figure 6 - Information on "What electrical cable to use?"
Source: students' report.

Thus, with the intention of finding the relationship between the voltage drop and the cable gauge, the group decided to observe the data of the table presented in Figure 6 from the Excel software, so that, observing the data dispersion graph and the trend line (Figure 7), they decide which is the behavior of the data. Therefore, the information provided by the teacher and the interpretative signs produced by the students from the software constitute immediate interpreters, considering that they produce in the students first impressions regarding the investigated phenomenon.

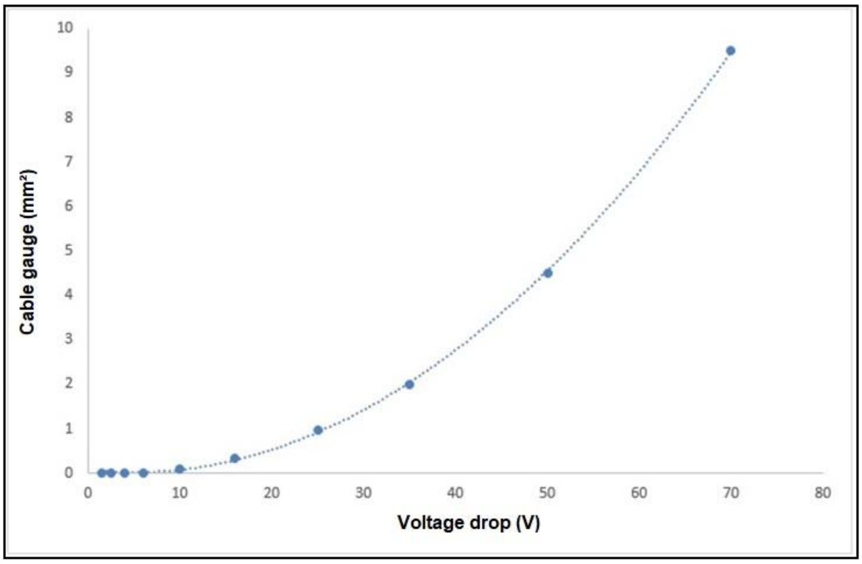


Figure 7 - Scatter plot and data trend line
Source: students' report.

By looking at the graph, the students came into contact with a sign that represents something rather than something else, and from this sign they decided ways to solve the problem under study. Santaella (2012, p. 15) states that "signs can only refer to something because, in some way, this something they denote is represented within the sign itself" and in this activity, specifically, the students highlighted what was represented (the drop of tension in relation to the cable gauge).

The mathematical procedures adopted by the group to adjust the quadratic function are shown in Figure 8.

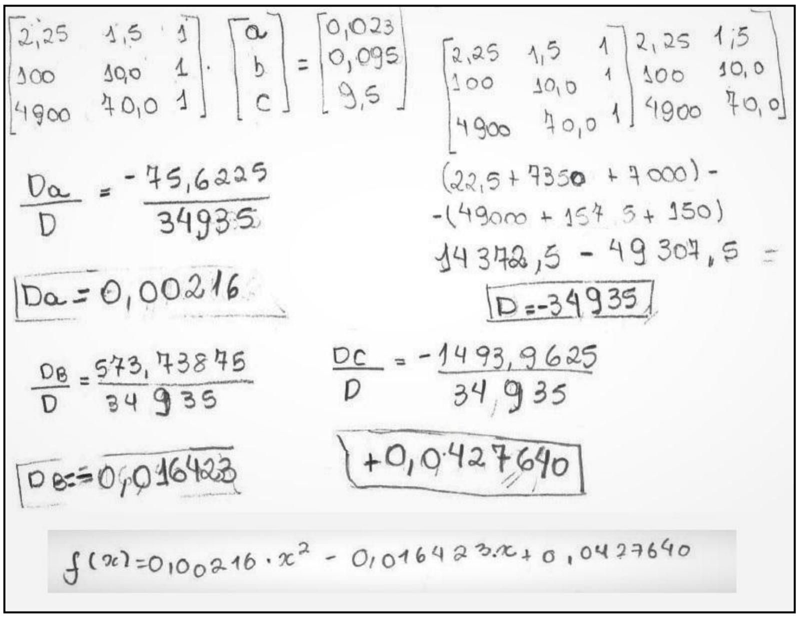


Figure 8 - Mathematical procedures adopted by the group to adjust the quadratic function
Source: students' report.

After obtaining the model f(x)=0.00216x²-0.016423x+0.042764, where f represents the value of the voltage drop of the electric current and x represents the electric cable gauge, the students decided to make a graphical analysis of the function in order to know the behavior of the function. For this, they made use of the Excel software. The graphical representation of the function is presented in Figure 9.

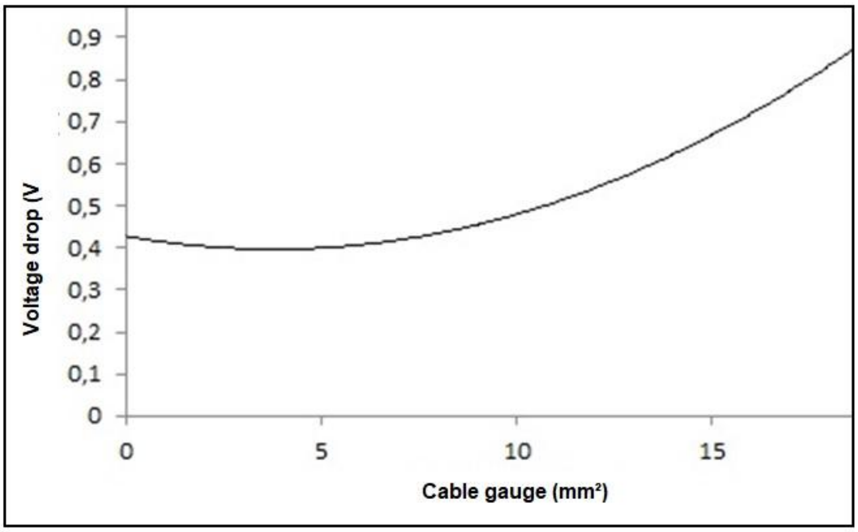


Figure 9 - Graphical representation of function $\boldsymbol{f(x) = 0,00216x^2 - 0,016423x + 0,42769}$
Source: students' report.

The production of this interpretative sign (graphic representation of the function) reveals the personal experiences of the students in order to understand the situation, at this moment still under construction, in relation to the possibilities for obtaining answers to the question formulated.

After that, considering that the purpose of this activity was to explore with the students the application of mathematical theory, they were introduced to the theory of calculating the maximum and minimum of a function through the use of the analysis of the derived function (Figure 10) in order to, from the model, answer the problem situation determined at the beginning of the activity: For the assembly of an electrical extension (for residential use) which would be the electrical cable gauges that would present, respectively, the smallest and the largest voltage drop?

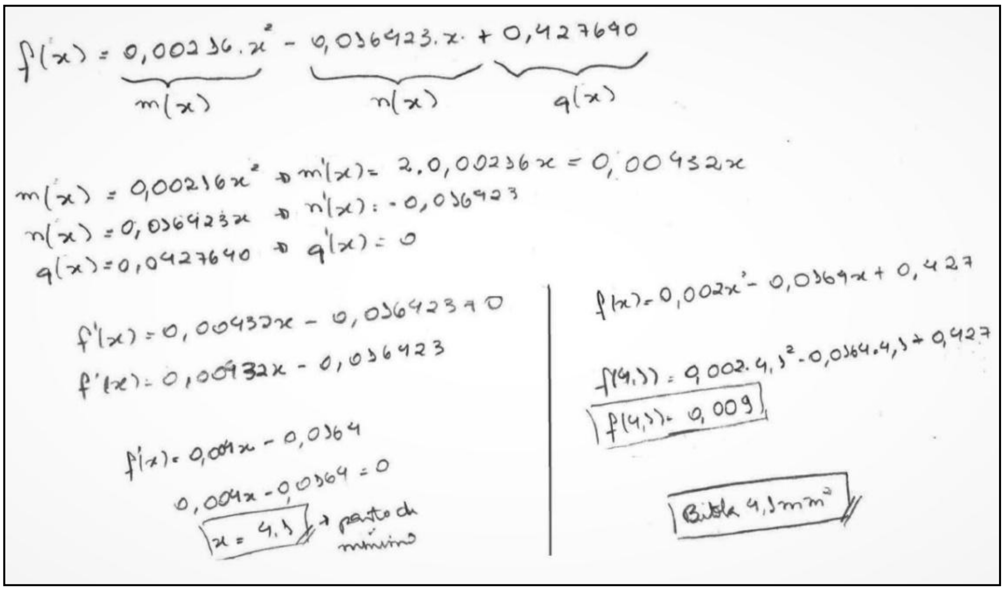


Figura 10 - Derivative analysis of function $\boldsymbol{f(x) = 0,00216x^2 - 0,016423x + 0,42769}$
Source: students' report.

Based on the analysis of the derived function, the group answers that the cable that would present the lowest voltage drop is the one with a gauge of 4.1 mm², as shown in Figure 10.

From the point of view of the mathematical development of the activity, the model that represents the phenomenon and the analysis of its derived function (f^' (x)=0.00432-0.016423), probably corresponds to a final interpreter. In fact, as stated by Peirce (2015, p. 164), the model could constitute "a true interpretation" which, in this case, satisfies and is sufficient to obtain the answers to the questions related to the choice of cables.

As a follow-up activity, more specifically a model exploration activity, the focus was, from the beginning, on the mathematical structure underlying the model developed by students.

Thus, regardless of the way students chose to define the mathematical model, in this case the use of the Cramer Rule to adjust a polynomial high school function, the intention of the activity was, after the model had been developed, to explore how to define the maximum (or minimum) value of this type of function.

Activity 3: Water loss in the water distribution process

The theme chosen by the group for this activity was the loss of water in the process of water distribution and the motivations for choosing the theme, according to the group's report, were: the worldwide problems of water waste; the ease with which the group obtained data on the distribution and loss of water in the city where they live, considering that one of the members works in the company responsible for water distribution in this region; and the fact that the city is considered one of the most problematic cities in the region with regard to the loss of water in distribution, according to data extracted from SANEPAR-PR documents.

The problem situation determined by the group was to elaborate a mathematical model that allows predictions about the loss of water in the municipality over the years.

In order to respond to the initial situation proposed, the group, based on data provided by SANEPAR, prepared Table 3.

Table 3 - Data collected by the group

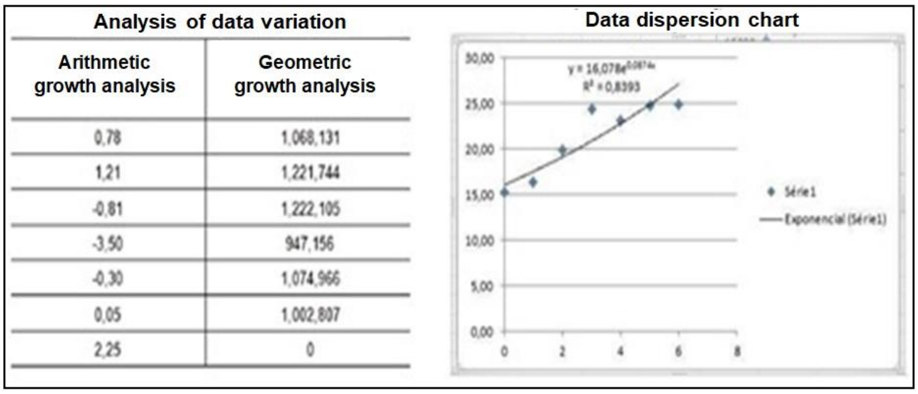
Average annual water loss in Urai-PR		
Year	*Average Annual Losses*	*Average Annual Losses in thousand m³*
2010	15297,0474	15,30
2011	16339,24873	16,34
2012	19962,37477	19,96
2013	24396,12128	24,40
2014	23106,93438	23,11
2015	24839,15784	24,84
2016	24908,88119	24,91

Source: students' report.

Having the data tabulated, the group carried out two analyses in order to know the behavior of the water loss in relation to the year and to define goals for solving the problem situation. In addition, they decided to analyze the dispersion of the data in the Geogebra software to visualize this behavior. The records of these analyses are presented in Figure 11.

In this case, the signs give evidence that the students understand that for the visualization of the points corresponding to the water loss in the distribution process, a table-like spatial arrangement is adequate (Table 3).

In order to identify if the growth of the data was based on a constant, the students chose to make an arithmetic analysis (subtracting one year's waste from their previous one) and geometric analysis (dividing one year's waste from their previous one) of the variation of the data, observing that the geometric behavior was closer to constant 1 (as shown in Figure 11). The group inferred that this was an exponential and non-linear behavior and so the students decided to determine a model of type $F(x) = a + b.e^{x.k}$, being F the volume of water lost (in thousand m³) and x the time (in years) from 2010.



Analysis of data variation

Arithmetic growth analysis	Geometric growth analysis
0,78	1,068,131
1,21	1,221,744
-0,81	1,222,105
-3,50	947,156
-0,30	1,074,966
0,05	1,002,807
2,25	0

Figure 11 - Records of data behavior analysis
Source: students' report.

The assumption made by the students was that the maximum volume lost in distribution will never exceed 40% of the total volume distributed. The hypothesis was elaborated due to a criterion imposed by the water distribution company that the loss in the distribution process should never exceed 40% of the total distributed, even considering the waste of the users and the problems in the pipes. Thus, this value of 40% was considered, by the group, the value of the asymptote of the function to be adjusted.

Thus, both the table in which the students analyzed the growth of the function and the graph were, at first, for this group of students, immediate interpreter signs capable of transmitting important information to the interpreters regarding the phenomenon (water loss in distribution). This information, to a certain extent, led to the group's research leading the interpreters to have some reactions (dynamic interpreters).

The mathematical procedures adopted by the group to adjust the exponential function are presented in Figure 12.

To obtain the values for the function:

$$F(x) = a + b.e^{(x.k)}$$

The values were replaced to then identify the values of the unknowns present in the function:

$$F(x) = 23{,}912 + b.e^{(x.k)}$$
$$15{,}3 = 23{,}912 + b.e^{(x.k)}$$
$$\mathbf{b = -8{,}61}$$

$$16{,}34 = 23{,}912 - 8{,}61.e^{(1.k)}$$
$$16{,}34 - 23{,}912 = -8{,}61.e^{k}$$
$$-7{,}57 = -8{,}61.e^{k}$$
$$-7{,}57/-8{,}61 = e^{k}$$
$$0{,}87 = e^{k}$$
$$log.0{,}87 = log.e^{k}$$
$$-0{,}06 = k.log.e$$
$$-0{,}06/0{,}43 = k$$
$$\mathbf{k=-0{,}13}$$

Assigning the values of the unknowns results in a model for this function, which is:

$$F(x) = 23{,}912 - 8{,}61.e^{(-0{,}13*x)}$$

Since the objective was to find a model that can be used in relation to the year in which the water loss is to be investigated, we adapted the model that is related to the auxiliary variable, so that we obtain it:

$$F(x) = 23{,}912 - 8{,}61.e^{(-0{,}13*x)}$$
Being $x = a - 2010$
Where $\boldsymbol{a} = ano$

We have:
$$F(a) = 23{,}912 - 8{,}61.e^{[(-0{,}13.(a-2010)]}$$
$$F(a) = 23{,}912 - 8{,}61.e^{(-0{,}13a+261{,}3)}$$

Figure 12 - Procedures adopted by students to adjust exponential function
Source: students' report.

After determining model $F(x) = 23{,}912 - 8{,}61\,.\,e^{(-0{,}13x+261{,}3)}$, the group performed its validation by comparing the results obtained by the model with the values provided by the sanitation company. Considering the model obtained to be valid, the group used it to predict the loss of water in the next 5 years as of 2017 (Table 4).

The obtaining of the model and its subsequent interpretation and validation indicate final interpretative signs because, at this moment, the students' view is loaded with interpretation, search for explanation, analysis and generalization, in which the group will be able to interpret the data that correspond to the mathematical object (exponential function) according to laws and mathematical concepts.

Table 4 - Forecast on water loss in Urai-PR

Year	*Water loss forecast (in thousand m³)*
2017	20,44
2018	20,86
2019	21,23

2020	21,56
2021	21,85
2022	22,10

Source: students' report.

In order to apply their knowledge of maximum and minimum of one function, given the context in which the activity was developed, the students determined a second problem: considering the mathematical model defined, in which year will occur, or occurred, the least loss of water in the process of water distribution in the city?

To answer this question, the students decided to analyze the function derived from the model obtained (f^' (x)=1,1193 .e^((-0,13.x+261,3))). For this, the students made use of two derivation rules that had already been studied in previous classes: the product derivative rule and the chain rule for composite function derivatives. The procedures performed by the students are described in Figure 13.

With the analysis of the derivative function, the group concludes that 2010 was the year in which the smallest water loss in the city water distribution occurred. At this time, this is the final interpreter of the group for this situation.

However, at a given moment, it is possible to notice that the group mistakenly considered that ln0=0. Besides being inadequate, the analysis of the derived function presented by the group shows a fragility in relation to the concept of derived function and what this means when applied in the study of the phenomenon.

Applications of the derivative in the function $f(x) = 23,912 - 8,61.e^{-0,13x+261,3}$

First, we calculate the derivative of $f(x)$

$$f(x) = 23,912 - 8,61.e^{-0,13x+261,3}$$

(23,912 is a constant, therefore its derivative is zero and we do not consider)

So

$$f'(x) = g'(x).h(x) + g(x).h'(x)$$

Being

$$g(x) = -8,61$$
$$g'(x) = 0$$
$$h(x) = e^{-0,13x+261,3}$$

Where

$$h'(x) = e^{u}.u'$$
$$u = -0,13x + 26,3$$
$$u' = -0,13$$
$$h'(x) = -0,13.e^{-0,13x+261,3}$$

$$F'(x) = 0.e^{-0,13x+261,3} + (-8,61.-0,13.e^{-0,13x+261,3})$$
$$f'(x) = 1,1193.e^{-0,13x+261,3}$$

When $f'(x)$ is equal to zero, we find the critical point of function:

$$f'(x) = 1,1193.e^{-0,13x+261,3}$$

$$1,1193.e^{-0,13x+261,3} = 0$$

$$e^{-0,13x+261,3} = \frac{0}{1,1193}$$

$$-0,13x + 261,3.\ln e = \ln 0$$

$$x = \frac{261,3}{0,13}$$

$$x = 2010$$

Therefore, 2010 is the critical point of the function.

Figure 13 - Analysis of the derived function of the model obtained
Source: Students' report

During the classes, the group delivered a new version of the report containing a correction regarding the application of the derivative in the model obtained. This correction is presented in Figure 14.

With the course of the classes it was possible to better understand the behavior of the function. Because it is an exponential function, it is growing all the time. Therefore, since its behavior is totally increasing, it is possible to conclude that the function has no critical point, nor inflexion point (which is the moment when the curvature of the function is modified). Therefore, the previous calculations are wrong. This makes sense for our model since, within the data obtained, there was not a time when the loss of water decreases over the years, considering that the loss is always greater over the years.

Figure 14 - Correction of the group regarding the analysis of the derivative function of the model
Source: students' report.

From the justification presented by the group we have evidence that, with the progress of classes, a new semiosis was triggered by students and the final interpreter generated in the activity (in 2010 - first year in which the water loss was analyzed - there was minimal loss in distribution), has generated new interpreters that have resulted in another understanding of the phenomenon by the students (as their behavior is totally increasing, it is concluded that the function has no critical point, nor inflexion point, that is, it is not possible to determine a year in which there was minimal loss), which is in line with what Almeida e Silva states (2017, p. 218), that "mathematical modeling activities trigger semiosis and, semiosis performs knowledge construction".

Like the previous one, the problem situation to be explored in this activity was determined by the students themselves. However, unlike the second activity, which focused on the mathematical structure of the model to be obtained by them, in this third activity the aim was to get students to think about a variety of situations in which they could use the model obtained, characteristic of a model application activity as defined by Ärlebäck and Doerr (2015). In the case of this activity, students used the same mathematical structure as the model obtained in activity 1 (Internet Users in Brazil), thus expanding the contexts of application of the exponential function of asymptotic behavior.

Discussion and results

The interlocutions between mathematical modeling and Peircean semiotics have been the subject of interest of teachers/researchers in the area (ALMEIDA E SILVA, 2017; ALMEIDA, 2010). In this article, we particularly direct our attention to the use of interpretative signs by students when developing a sequence of mathematical modeling activities, in line with what characterizes Lesh et al. (2003). The sequence of activities that we present had the objective of introducing the concept of extreme values of a real function in the subject.

The development of the sequence of activities allows the organization and elaboration of signs, showing what the students know about mathematics and the phenomenon.

The analysis of the signs produced or used by students allows us to infer that a sequence of mathematical modeling activities provides moments of exploration and application of models, either when students feel the need to refine their models to better represent the phenomenon explored or even when they use the models obtained in different contexts, which shows that the previous knowledge of the students and their experiences are fundamental to the resolution of the situations explored.

In the development of the sequence, the immediate interpreters, according to the classification of Peircean semiotics, are related to the first impressions shown by the interpreters when they first come into contact with the situations to be explored.

Dynamic interpreter signs, in turn, are related to the interpreters' reactions to their immediate interpreters. In addition, these dynamic interpreters are related to the search for information that students seek in order to start studying situations with the definition of the problem to be solved, with the existence of something to be explored.

The final interpretive signs in the sequence of mathematical modeling activities are related to obtaining and deducing the mathematical model and obtaining the mathematical results and their validation, taking into consideration aspects related to mathematics and the phenomenon.

The first activity of the sequence - Internet users in Brazil - was developed before the first contact of the students with the concept of extreme of a function. It constitutes in the sequence a warm-up activity, as characterized by Lesh et al. (2003), aiming at involving students in the action of constructing a mathematical model and discussing its relevance for addressing a non-mathematical problem.

Ärlebäck and Doerr (2015) state that warm-up activities should confront students with the initial need to develop a model in order to make sense of a previously defined situation. Considering this assertion, students were offered a problem that needed the deduction of a mathematical model in order to obtain a solution, moreover, for such a solution, previously studied mathematical contents needed to be applied and explored. In the context of the discipline of Differential and Integral Calculus, it provided the construction of an asymptotic model. Thus, although the function does not have a maximum value, it was possible to discuss the meaning of asymptotic in the study of a phenomenon.

In this sense, the use and handling of data and the construction of the mathematical model come surrounded by interpretative signs that do not yet provide contact with the concept of extreme value of a function but help in the interpretation of an asymptotic model.

As discussed in the literature, it is common for a warm-up activity to cause certain discomfort to students (LESH et al., 2003; ÄRLEBÄCK; DOERR, 2018). In the first activity, such discomforts could be identified, for example, at the time when students deemed it necessary to survey hypotheses for the resolution of the activities and the need for further research to understand the situation under study. The overcoming of these discomforts was mediated by the action of the interpretative signs, in line with what Almeida e Silva (2017) advocates and may be an indication of semiosis in this activity.

In the second activity of the sequence of mathematical modeling activities, in the phase of follow-up activities, although the topic was suggested by the researcher and he also took the data into the classroom, the students built the mathematical model without the intervention of the teacher.

In this activity with the theme Optimization of the dimension of electrical cables the intention was to explore the application of extreme values of a function in everyday situations. Thus, after getting acquainted with the theme presented (Choosing the energy cable according to the gauge) and defining goals for its resolution, the group was introduced to the theory of the extreme value of a function through the use of the analysis of the derived function.

Such maximums and minimums would be used to answer the question: For the assembly of an electrical extension (for residential use) what would be the electrical cable gauges that would present, respectively, the lowest and the highest voltage drop?

The students already knew the rules of derivation, so when they faced the need to analyze the derivation, immediate interpreters were imperceptible. Therefore, unlike the warm-up activity, in the follow-up activity, specifically in the exploration activity of a model, what could be evidenced were dynamic interpretative signs. Such interpreters are related, mainly, to the actions of the students regarding the calculations and analysis necessary to solve the problem.

This activity also provided the students with the analysis of different representations (algebraic and graphic, for example) for the same situation, allowing them to identify structural similarities and differences between and among such representations.

Finally, in the third activity of the sequence of activities, Water loss in the distribution in the city of Urai-PR, the problem defined by the group was to elaborate a model that allows the prediction of water loss over the years in the city.

The development of this activity required the application of the mathematical model in different contexts. In this activity, to obtain one of the mathematical models, the students used an exponential function of asymptotic behavior, similar to what they used in the warm-up activity.

However, the model constructed in this follow-up activity required adaptations in relation to the asymptotic model they were already familiar with, enabling them to further understand and refine their language, although analogies could, at times, provide the production of more conclusive interpretative signs in relation to the minimum value of a function. In this sense, the final interpreter in this activity can be identified with respect to an extreme value of a function. In fact, in this activity students have strong evidence that they know what extreme points of a function are and, from the point of view of the problem under study, are able to explain the meaning of the point of inflection.

From the analysis of this activity, we can also infer that the use of mathematics that students have made during the development of the activity is anchored in their previous experiences, or collateral knowledge as defined by Peirce (2015), either in their experiences with mathematics concepts and tools, or in their practical experiences with mathematical modeling (ALMEIDA, 2018).

The interpretative signs produced by students during the development of mathematical modeling activities suggest that the development of sequences of modeling activities can provide students with experience in dealing with the construction and interpretation of models and provide them with the contact of different nuances of a mathematical object, particularly the derivative and its use to analyze phenomena, with respect to the existence of maximum and minimum values.

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